MAT124 MATHEMATICS II

Calculating Flux Integrals, Vector Calculus: Gradient, Divergence, and Curl

Outline

Calculating Flux Integrals

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1. Parametric Surfaces

If $\mathcal{S}$ is a parametric surface given by $\mathbf{r} = \mathbf{r}(u, v)$ for (u, v) in domain D in the uv-plane:

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Calculating Flux Integrals

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- The area element on $\mathcal{S}$ is $dS = |\mathbf{n}|\, du\, dv$

Calculating Flux Integrals

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- Accordingly, the vector area element $d\mathbf{S}$ becomes:

  $$d\mathbf{S} = \hat{\mathbf{N}}\, dS = \pm \left(\frac{\mathbf{n}}{|\mathbf{n}|} \right) |\mathbf{n}|\, du\, dv = \pm \mathbf{n}\, du\, dv$$

  where the sign must be chosen to reflect the desired orientation on $\mathcal{S}$ (e.g., upward/downward, outward/inward).

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Flux of $\mathbf{F}$ through a Parametric Surface

The flux of $\mathbf{F}(x, y, z)$ through $\mathcal{S}$ is given by

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \pm \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) du\, dv$$

Calculating Flux Integrals

2. Implicit Surfaces

Let $\mathcal{S}$ be a smooth, oriented surface with a one-to-one projection onto a domain D in the xy-plane, with an equation of the form $\mathbf{G}(x, y, z) = 0$.

Calculating Flux Integrals

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- Depending on the orientation of $\mathcal{S}$, the unit normal $\hat{\mathbf{N}}$ can be written as:

$$\hat{\mathbf{N}} = \pm \frac{\nabla G}{|\nabla G|}$$

Calculating Flux Integrals

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Choosing the Sign

The sign must be chosen to give $\mathcal{S}$ the desired orientation. If $G_3 > 0$ and we want the positive side of $\mathcal{S}$ to face upward, we use the $+$ sign.

Calculating Flux Integrals

Summary: Formulas based on Surface Representation

How we compute $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ depends on how $\mathcal{S}$ is defined:

- **1. Parametric Surface $\mathbf{r}(u, v)$:**

$$d\mathbf{S} = \pm \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) du\, dv$$

- **2. Implicit Surface $G(x, y, z) = 0$:**

$$d\mathbf{S} = \pm \frac{\nabla G(x, y, z)}{G_3(x, y, z)}\, dx\, dy$$

- **3. Explicit Surface $z = f(x, y)$:**

$$d\mathbf{S} = \pm \left(-\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k} \right) dx\, dy$$

Note: The ± sign is chosen to reflect the desired orientation (e.g., upward/downward, outward/inward).

Calculating Flux Integrals

Example: Explicit Surface

EXAMPLE

Find the flux of $\mathbf{F} = z\mathbf{i} + x^2\mathbf{k}$ **upward** through that part of the surface $z = x^2 + y^2$ lying above the square R defined by $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.

Calculating Flux Integrals

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Solution: Let $G(x, y, z) = z - x^2 - y^2 = 0$. We have $\nabla G = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$ and $G_3 = 1$.

Calculating Flux Integrals

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Since $G_3 > 0$, taking the $+$ sign gives the upward normal.
Thus, $d\mathbf{S} = (-2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k})\,dx\,dy$.

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The required flux is:

$$\iint_{\mathcal{S}} (z\mathbf{i} + x^2\mathbf{k}) \cdot d\mathbf{S} = \iint_R \left(-2x(x^2 + y^2) + x^2\right) dx\,dy$$

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Oriented Surfaces and Flux Integrals

Calculating Flux Integrals

EXAMPLE

Find the flux of $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + 4\mathbf{k}$ upward through $\mathcal{S}$, where $\mathcal{S}$ is the part of the surface $z = 1 - x^2 - y^2$ lying in the first octant of 3-space.

Oriented Surfaces and Flux Integrals

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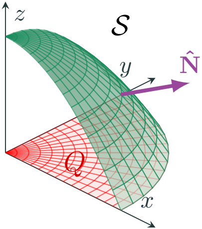

Solution. The vector area element corresponding to the upward normal on $\mathcal{S}$ is

$$d\mathbf{S} = \left(-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}\right) dx\, dy = (2x\mathbf{i}+2y\mathbf{j}+\mathbf{k})dx\, dy.$$

Oriented Surfaces and Flux Integrals

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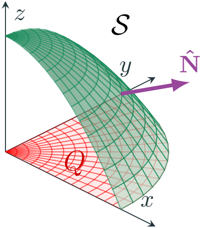

The projection of $\mathcal{S}$ onto the xy-plane is the quarter-circular disk Q given by $x^2 + y^2 \leq 1$, $x \geq 0$, and $y \geq 0$.

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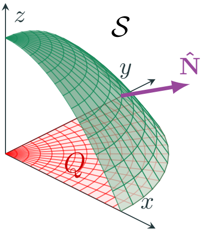

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Thus, the flux of $\mathbf{F}$ upward through $\mathcal{S}$ is

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_Q (2xy - 2xy + 4)\, dx\, dy = 4 \times (\text{area of } Q) = \boxed{\pi}.$$

Calculating Flux Integrals

Example: Parametric Surface

EXAMPLE

Find the flux of $\mathbf{F} = \frac{2x\mathbf{i}+2y\mathbf{j}}{x^2+y^2} + \mathbf{k}$ **downward** through the surface defined parametrically by $\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u^2\,\mathbf{k}$, where $0 \le u \le 1$, $0 \le v \le 2\pi$.

Calculating Flux Integrals

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Solution. First we calculate $d\mathbf{S}$:

$$\frac{\partial \mathbf{r}}{\partial u} = \cos v\,\mathbf{i} + \sin v\,\mathbf{j} + 2u\,\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial v} = -u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j}$$

Calculating Flux Integrals

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$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = -2u^2\cos v\,\mathbf{i} - 2u^2\sin v\,\mathbf{j} + u\,\mathbf{k}$$

Since $u \ge 0$, the k-component is positive (upward). We want a **downward** normal, so we multiply by -1:

$$d\mathbf{S} = (2u^2\cos v\,\mathbf{i} + 2u^2\sin v\,\mathbf{j} - u\,\mathbf{k})\,du\,dv$$

Calculating Flux Integrals

Example: Parametric Surface (cont.)

Solution (cont). On the surface $\mathcal{S}$, substituting $x = u\cos v$ and $y = u\sin v$, we have $x^2 + y^2 = u^2$. The vector field evaluates to:

$$\mathbf{F} = \frac{2u\cos v\,\mathbf{i} + 2u\sin v\,\mathbf{j}}{u^2} + \mathbf{k} = \frac{2\cos v}{u}\mathbf{i} + \frac{2\sin v}{u}\mathbf{j} + \mathbf{k}$$

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Now, take the dot product $\mathbf{F}\cdot d\mathbf{S}$:

$$\begin{aligned}\mathbf{F}\cdot d\mathbf{S} &= \left[\left(\frac{2\cos v}{u}\right)(2u^2\cos v) + \left(\frac{2\sin v}{u}\right)(2u^2\sin v) + (1)(-u)\right] du\,dv\\ &= (4u\cos^2 v + 4u\sin^2 v - u)\,du\,dv\\ &= (4u - u)du\,dv = 3u\,du\,dv\end{aligned}$$

Calculating Flux Integrals

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The downward flux of $\mathbf{F}$ through $\mathcal{S}$ is:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^1 (3u)\,du\,dv$$

Calculating Flux Integrals

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$$\begin{aligned} \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^1 (3u)\,du\,dv \\ &= \int_0^{2\pi} dv \left[\frac{3u^2}{2}\right]_0^1 = 2\pi\left(\frac{3}{2}\right) = \boxed{\mathbf{3}\pi} \end{aligned}$$

Vector Calculus: Gradient, Divergence, and Curl

Gradient, Divergence, and Curl

First-order information about the rate of change of a three-dimensional scalar field,

$$f(x, y, z),$$

is contained in the three first partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$.

The gradient,

$$\boxed{\operatorname{grad} f(x, y, z) = \nabla f(x, y, z) = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k},}$$

collects this information into a single vector-valued derivative of f.

Gradient, Divergence, and Curl

First-order information about the rate of change of the vector field

$$\mathbf{F}(x, y, z) = F_1(x, y, z)\,\mathbf{i} + F_2(x, y, z)\,\mathbf{j} + F_3(x, y, z)\,\mathbf{k}$$

is contained in nine first partial derivatives:

$$\begin{array}{ccc} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} & \dfrac{\partial F_1}{\partial z} \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} & \dfrac{\partial F_2}{\partial z} \\ \dfrac{\partial F_3}{\partial x} & \dfrac{\partial F_3}{\partial y} & \dfrac{\partial F_3}{\partial z} \end{array}$$

Gradient, Divergence, and Curl

Divergence and Curl

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z},$$

Gradient, Divergence, and Curl

Divergence and Curl

$$
\begin{aligned}
\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}, \\
\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k},
\end{aligned}
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Gradient, Divergence, and Curl

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&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.
\end{aligned}$$

Gradient, Divergence, and Curl

Divergence and Curl

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\operatorname{div}\mathbf{F} &= \nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z},\\
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The above definition uses the vector differential operator

$$\nabla = \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k}.$$

Gradient, Divergence, and Curl

Example

Find the divergence and curl of the vector field

$$\mathbf{F} = xy\,\mathbf{i} + (y^2 - z^2)\,\mathbf{j} + yz\,\mathbf{k}.$$

Gradient, Divergence, and Curl

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Find the divergence and curl of the vector field

$$\mathbf{F} = xy\,\mathbf{i} + (y^2 - z^2)\,\mathbf{j} + yz\,\mathbf{k}.$$

Solution: We have

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2 - z^2) + \frac{\partial}{\partial z}(yz) = y + 2y + y = 4y.$$

Gradient, Divergence, and Curl

Example

Find the divergence and curl of the vector field

$$\mathbf{F} = xy\,\mathbf{i} + (y^2 - z^2)\,\mathbf{j} + yz\,\mathbf{k}.$$

Solution: We have

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2 - z^2) + \frac{\partial}{\partial z}(yz) = y + 2y + y = 4y.$$

$$\begin{aligned}\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & y^2 - z^2 & yz \end{vmatrix} \\ &= \left[\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(y^2 - z^2)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(yz)\right]\mathbf{j} \\ &\quad + \left[\frac{\partial}{\partial x}(y^2 - z^2) - \frac{\partial}{\partial y}(xy)\right]\mathbf{k} = 3z\,\mathbf{i} - x\,\mathbf{k}.\end{aligned}$$

Gradient, Divergence, and Curl

The divergence and curl of a two-dimensional vector field can also be defined: if

$$\mathbf{F}(x,y) = F_1(x,y)\,\mathbf{i} + F_2(x,y)\,\mathbf{j},$$

then

$$\operatorname{div}\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}, \qquad \operatorname{curl}\mathbf{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

Note that the curl of a two-dimensional vector field is still a 3-vector and is perpendicular to the plane of the field.

Gradient, Divergence, and Curl

Example

Find the divergence and curl of

$$\mathbf{F} = xe^y\,\mathbf{i} - ye^x\,\mathbf{j}.$$

Solution:

$$\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(xe^y) + \frac{\partial}{\partial y}(-ye^x) = e^y - e^x,$$

$$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \left(\frac{\partial}{\partial x}(-ye^x) - \frac{\partial}{\partial y}(xe^y)\right)\mathbf{k} = -(ye^x + xe^y)\mathbf{k}.$$

Gradient, Divergence, and Curl

Interpretation of the Divergence

The value of the divergence of a vector field $\mathbf{F}$ at point P is, loosely speaking, a measure of the rate at which the field diverges or spreads away from P. This spreading away can be measured by the flux out of a small closed surface surrounding P. For instance, $\operatorname{div}\mathbf{F}(P)$ is the limit of the flux per unit volume out of smaller and smaller spheres centred at P.

Theorem. *The divergence as flux density*

If $\hat{\mathbf{N}}$ is the unit outward normal on the sphere $\mathcal{S}_\varepsilon$ of radius ε centred at point P, and if $\mathbf{F}$ is a smooth three-dimensional vector field, then

$$\operatorname{div}\mathbf{F}(P) = \lim_{\varepsilon\to 0^+} \frac{3}{4\pi\varepsilon^3} \iint_{\mathcal{S}_\varepsilon} \mathbf{F}\cdot\hat{\mathbf{N}}\,dS.$$

Gradient, Divergence, and Curl

Physical Interpretation of Divergence

Physically, the divergence of a vector field $\mathbf{F}$ at a point P measures the net tendency of the fluid to **flow outward** from P. It is the *flux density* or *expansion rate*.

Gradient, Divergence, and Curl

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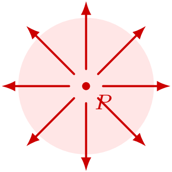
$\operatorname{div} \mathbf{F} > 0$

Point Source (Expands)



Fluid is created or expanding at P.

Gradient, Divergence, and Curl

Physical Interpretation of Divergence

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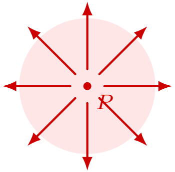
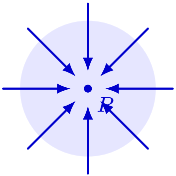


Fluid is created or expanding at P.

$\operatorname{div}\mathbf{F} < 0$

Sink (Compresses)



Fluid is destroyed or compressing.

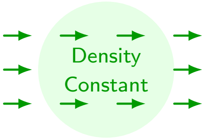
Gradient, Divergence, and Curl

Physical Interpretation of Divergence

Physically, the divergence of a vector field $\mathbf{F}$ at a point P measures the net tendency of the fluid to **flow outward** from P. It is the *flux density* or *expansion rate*.

$\operatorname{div}\mathbf{F} = 0$

Incompressible



What goes in, comes out exactly.

Gradient, Divergence, and Curl

Physical Interpretation of Divergence

Physically, the divergence of a vector field $\mathbf{F}$ at a point P measures the net tendency of the fluid to **flow outward** from P. It is the *flux density* or *expansion rate*.

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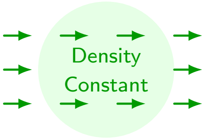
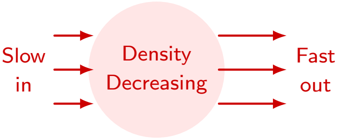
Incompressible



What goes in, comes out exactly.

$\operatorname{div}\mathbf{F} > 0$

Accelerating Flow



More flows out than in.

Gradient, Divergence, and Curl

Why is $\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F}$?

To understand why divergence is the sum of partial derivatives, consider a tiny 2D box of size $\Delta x \times \Delta y$ around point (x, y). Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ be the general flow.

Gradient, Divergence, and Curl

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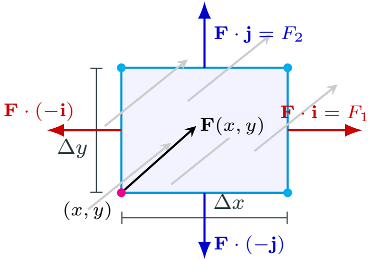

Gray arrows represent the general field $\mathbf{F}$. We only extract the normal (perpendicular) components.

Gradient, Divergence, and Curl

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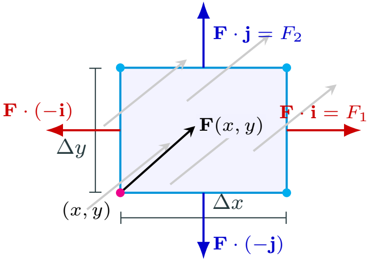

Gray arrows represent the general field $\mathbf{F}$. We only extract the normal (perpendicular) components.

Flux across an edge is $\mathbf{F} \cdot \hat{\mathbf{n}} \times$ (length). For vertical edges ($\pm\mathbf{i}$), we extract $\pm F_1$:

$$\text{Net}_x \approx \left[F_1(x + \Delta x, y) - F_1(x, y)\right] \Delta y$$

Gradient, Divergence, and Curl

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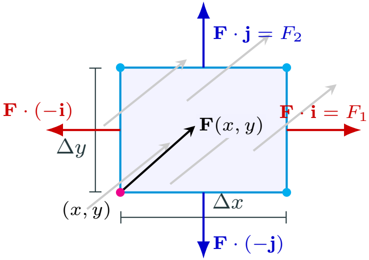

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$$\text{Net}_x \approx [F_1(x + \Delta x, y) - F_1(x, y)]\, \Delta y$$

Divide and multiply by Δx:

$$\text{Net}_x \approx \left(\frac{\partial F_1}{\partial x}\Delta x\right)\Delta y = \frac{\partial F_1}{\partial x}\underbrace{(\Delta x \Delta y)}_{\text{Area } \Delta A}$$

Gradient, Divergence, and Curl

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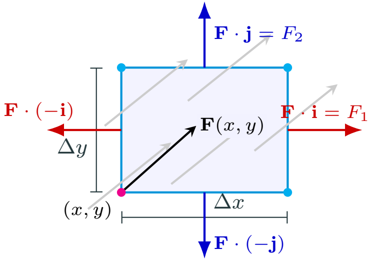

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Divide and multiply by Δx:

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Similarly, horizontal edges ($\pm\mathbf{j}$) extract F_2. Summing them up:

$$\text{Total Flux} \approx \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right)\Delta A$$

Gradient, Divergence, and Curl

Why is $\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F}$?

The Notation $\nabla \cdot \mathbf{F}$

Dividing the total flux by the area (or volume in 3D) yields the exact definition of the dot product with the del operator:

$$\text{Flux Density} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = \nabla \cdot \mathbf{F}$$

or in three dimensions:

$$\text{Flux Density} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \nabla \cdot \mathbf{F}$$

Gradient, Divergence, and Curl

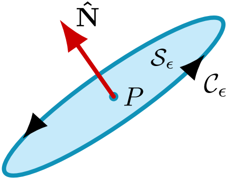
Interpretation of the Curl

Roughly speaking, $\operatorname{curl}\mathbf{F}(P)$ measures the extent to which the vector field $\mathbf{F}$ swirls around P.

Theorem. *The curl as circulation density*

If $\mathbf{F}$ is a smooth vector field and $\mathcal{C}_\varepsilon$ is a circle of radius ε centred at point P and bounding a disk $\mathcal{S}_\varepsilon$ with unit normal $\hat{\mathbf{N}}$ (and orientation inherited from $\mathcal{C}_\varepsilon$), then

$$\lim_{\varepsilon\to 0^+} \frac{1}{\pi\varepsilon^2} \oint_{\mathcal{C}_\varepsilon} \mathbf{F}\cdot d\mathbf{r} = \hat{\mathbf{N}}\cdot\operatorname{curl}\mathbf{F}(P).$$



Gradient, Divergence, and Curl

Physical Interpretation of the Curl

Roughly speaking, $\operatorname{curl}\mathbf{F}(P)$ measures the extent to which the vector field $\mathbf{F}$ **swirls (or rotates)** around P. (or how much that fluid tends to rotate around P.) Imagine placing a tiny *paddle wheel* in the fluid. The curl dictates how fast and in what direction the wheel will spin.

Gradient, Divergence, and Curl

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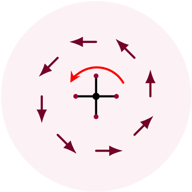
$\operatorname{curl}\mathbf{F} \neq \mathbf{0}$

Macroscopic Rotation (CCW)



The whole fluid rotates (e.g., a whirlpool). The wheel spins clearly.

Gradient, Divergence, and Curl

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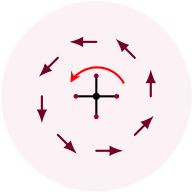
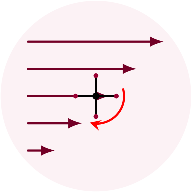


The whole fluid rotates (e.g., a whirlpool). The wheel spins clearly.

$\operatorname{curl}\mathbf{F} \neq \mathbf{0}$

Shear Flow (CW)



Flow is straight, but top is faster. Velocity difference spins the wheel CW!

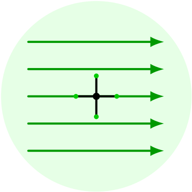
Gradient, Divergence, and Curl

Physical Interpretation of the Curl

Roughly speaking, $\operatorname{curl}\mathbf{F}(P)$ measures the extent to which the vector field $\mathbf{F}$ **swirls (or rotates)** around P. (or how much that fluid tends to rotate around P.) Imagine placing a tiny *paddle wheel* in the fluid. The curl dictates how fast and in what direction the wheel will spin.

$\operatorname{curl}\mathbf{F} = \mathbf{0}$

Irrotational (Uniform Flow)



The wheel is pushed equally from all sides. It moves, but does **not** spin.

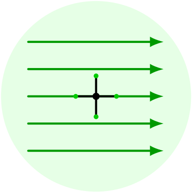
Gradient, Divergence, and Curl

Physical Interpretation of the Curl

Roughly speaking, $\operatorname{curl}\mathbf{F}(P)$ measures the extent to which the vector field $\mathbf{F}$ **swirls (or rotates)** around P. (or how much that fluid tends to rotate around P.) Imagine placing a tiny *paddle wheel* in the fluid. The curl dictates how fast and in what direction the wheel will spin.

$\operatorname{curl}\mathbf{F} = \mathbf{0}$

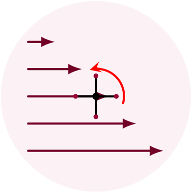
Irrotational (Uniform Flow)



The wheel is pushed equally from all sides. It moves, but does **not** spin.

$\operatorname{curl}\mathbf{F} \neq \mathbf{0}$

Reverse Shear Flow (CCW)



Bottom flows faster than top. The wheel is pushed from below and spins CCW!

Gradient, Divergence, and Curl

Why is $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F}$?

Instead of flow *out* of a small region (flux), let us look at the flow **along** the boundary of a tiny rectangle in the xy-plane. We take the tangential components $\mathbf{F} \cdot d\mathbf{r}$ counter-clockwise, as seen from the positive z-axis.

Gradient, Divergence, and Curl

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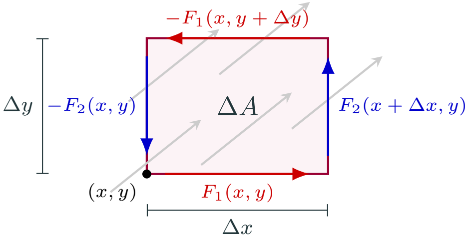

Gradient, Divergence, and Curl

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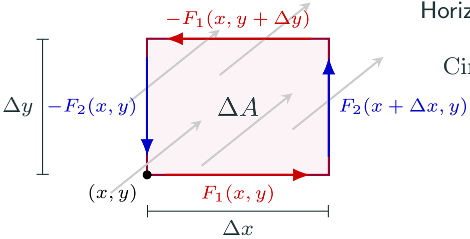
Instead of flow *out* of a small region (flux), let us look at the flow **along** the boundary of a tiny rectangle in the xy-plane. We take the tangential components $\mathbf{F} \cdot d\mathbf{r}$ counter-clockwise, as seen from the positive z-axis.



Horizontal paths: bottom and top

$$\mathrm{Circ}_x \approx F_1(x, y)\Delta x - F_1(x, y + \Delta y)\Delta x$$

$$\mathrm{Circ}_x \approx -\frac{\partial F_1}{\partial y}\Delta y\,\Delta x.$$

Gradient, Divergence, and Curl

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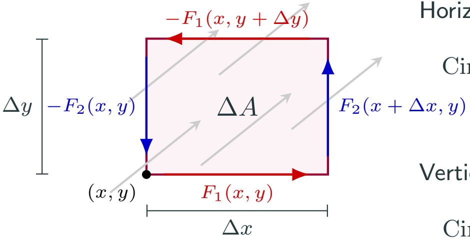
Instead of flow *out* of a small region (flux), let us look at the flow **along** the boundary of a tiny rectangle in the xy-plane. We take the tangential components $\mathbf{F} \cdot d\mathbf{r}$ counter-clockwise, as seen from the positive z-axis.



Horizontal paths: bottom and top

$$\mathrm{Circ}_x \approx F_1(x, y)\Delta x - F_1(x, y+\Delta y)\Delta x$$

$$\mathrm{Circ}_x \approx -\frac{\partial F_1}{\partial y}\Delta y\,\Delta x.$$

Vertical paths: right and left

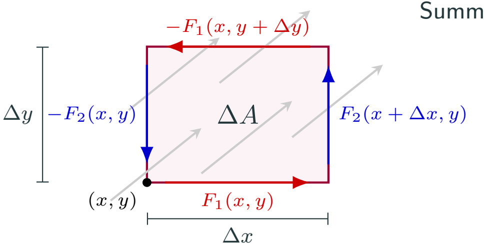
$$\mathrm{Circ}_y \approx F_2(x+\Delta x, y)\Delta y - F_2(x, y)\Delta y$$

$$\mathrm{Circ}_y \approx \frac{\partial F_2}{\partial x}\Delta x\,\Delta y.$$

Gradient, Divergence, and Curl

Why is $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F}$?

Instead of flow *out* of a small region (flux), let us look at the flow **along** the boundary of a tiny rectangle in the xy-plane. We take the tangential components $\mathbf{F} \cdot d\mathbf{r}$ counter-clockwise, as seen from the positive z-axis.



Summing them gives the total circulation:

$$\text{Circ} \approx \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \underbrace{\Delta x \Delta y}_{\Delta A}.$$

Gradient, Divergence, and Curl

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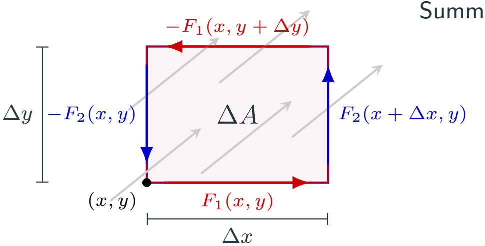
Instead of flow *out* of a small region (flux), let us look at the flow **along** the boundary of a tiny rectangle in the xy-plane. We take the tangential components $\mathbf{F} \cdot d\mathbf{r}$ counter-clockwise, as seen from the positive z-axis.



Summing them gives the total circulation:

$$\text{Circ} \approx \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \underbrace{\Delta x \Delta y}_{\Delta A}.$$

The $\mathbf{k}$-component of Curl

Dividing by the area gives the circulation density around the z-axis:

$$(\operatorname{curl}\mathbf{F}) \cdot \mathbf{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

This is exactly the $\mathbf{k}$-component of $\nabla \times \mathbf{F}$.

Gradient, Divergence, and Curl

Curl in Three Dimensions

In the previous slide, we looked at circulation around a tiny rectangle in the xy-plane. That circulation measures rotation around the z-axis.

$$(\operatorname{curl}\mathbf{F})\cdot\mathbf{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

Gradient, Divergence, and Curl

Curl in Three Dimensions

But in three dimensions, a fluid can rotate around any axis. Therefore, we must look at circulation around tiny rectangles in three coordinate planes:

rectangle in the yz-plane ⟶ rotation around the x-axis,
rectangle in the xz-plane ⟶ rotation around the y-axis,
rectangle in the xy-plane ⟶ rotation around the z-axis.

Gradient, Divergence, and Curl

Curl in Three Dimensions

But in three dimensions, a fluid can rotate around any axis. Therefore, we must look at circulation around tiny rectangles in three coordinate planes:

rectangle in the yz-plane $\longrightarrow$ rotation around the x-axis,
rectangle in the xz-plane $\longrightarrow$ rotation around the y-axis,
rectangle in the xy-plane $\longrightarrow$ rotation around the z-axis.

Thus, curl is a vector whose components are circulation densities:

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

Gradient, Divergence, and Curl

Curl in Three Dimensions

But in three dimensions, a fluid can rotate around any axis. Therefore, we must look at circulation around tiny rectangles in three coordinate planes:

$$\begin{aligned}
\text{rectangle in the } yz\text{-plane} \quad &\longrightarrow \quad \text{rotation around the } x\text{-axis},\\
\text{rectangle in the } xz\text{-plane} \quad &\longrightarrow \quad \text{rotation around the } y\text{-axis},\\
\text{rectangle in the } xy\text{-plane} \quad &\longrightarrow \quad \text{rotation around the } z\text{-axis}.
\end{aligned}$$

Thus, curl is a vector whose components are circulation densities:

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

Intuition

The curl vector points along the axis about which the vector field has the strongest local counterclockwise circulation. Its magnitude measures how strong that local rotation is.